Fractional Calculus and Applied Analysis
https://doi.org/10.1007/s13540-024-00284-2

Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications

ORIGINAL PAPER

Application of subordination principle to coefficient inverse problem for multi-term time-fractional wave equation

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Received: 28 February 2024 / Revised: 29 March 2024 / Accepted: 11 April 2024

Abstract
An initial-boundary value problem for the multi-term time-fractional wave equation on a bounded domain is considered. For the largest and smallest orders of the involved Caputo fractional time-derivatives, α and α_m, it is assumed $1 < \alpha < 2$ and $\alpha - \alpha_m \leq 1$. Subordination principle with respect to the corresponding single-term time-fractional wave equation of order α is deduced. Injectivity of the integral transform, defined by the subordination relation, is established. The subordination identity is used to prove uniqueness for a coefficient inverse problem for the multi-term equation, based on an analogous property for the related single-term one. In addition, the subordination relation is applied for deriving a regularity estimate.

Keywords Caputo fractional derivative · Time-fractional wave equation · Subordination principle · Coefficient inverse problem · Multinomial Mittag-Leffler function

Mathematics Subject Classification 26A33 · 33E12 · 35E15 · 35R11 · 35R30

1 Introduction

Time-fractional wave equations have attracted the attention of the scientists due to their ability for modeling of wave propagation in complex media, for instance, in the dynamical theory of linear viscoelasticity or for modeling the power-law attenuation when sound waves travel through inhomogeneous media [31]. In particular, the time-fractional wave equation in its simplest form, where the ordinary second derivative in time in the classical wave equation is substituted by a fractional Caputo derivative of order $1 < \alpha < 2$, governs the propagation of mechanical diffusive waves in viscoelastic media exhibiting a power-law creep [24].

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Published online: 29 April 2024

In the attempt to find more adequate models, the single fractional time derivative is often replaced by a discrete or continuous distribution of fractional time derivatives over the interval $(0, 2)$, see e.g. [2], Chapter 6. Such generalized fractional wave equations and other generalizations employing different memory kernels are a subject of extensive studies in the last decade. For example, generalized Cattaneo equations are discussed in [3, 12, 28], other generalized fractional wave equations are studied in [4, 31, 32], analytical solutions to initial-boundary value problems for multi-term fractional diffusion-wave equations are derived in [14]. Extensions of the regularity results in [30] concerning the fractional diffusion-wave equation with a single Caputo time-derivative can be found in [20] and in [34] for the multi-term diffusion and wave equations, respectively.

One possible way to study multi-term wave equations is by applying the subordination principle, which allows to construct solutions of complex evolution equations from the solutions of classical integer order equations, or simpler fractional order ones. The principle of subordination naturally emerges as a notion in physics (see the survey [11]) at the same time providing a useful tool in the study of the corresponding mathematical models, e.g. for establishing well-posedness, for deriving regularity estimates, or in the study of inverse problems. Subordination principle for various generalized fractional diffusion-wave equations is established in [4, 6, 8, 9, 28, 29, 35].

Inverse problems for fractional evolution equations are studied by many authors recently, see e.g. the review paper [17]. A standard inverse problem is to recover a coefficient in an elliptic operator acting in space in a diffusion or wave equation from a combination of initial data and over-specified boundary data. For such coefficient inverse problems we refer to [16, 22, 25].

In [25] the subordination relation between the time-fractional wave equation and a corresponding classical second order equation is applied to prove uniqueness to a coefficient inverse problem for the fractional one. In the present work we extend this result of [25] to an initial-boundary value problem for the multi-term time-fractional wave equation on a bounded domain. Based on the subordination relation between this multi-term equation and the corresponding single-term one, we derive the uniqueness property to a coefficient inverse problem for the multi-term fractional wave equation from the uniqueness result in [25].

In addition, as another application of the subordination principle, a regularity estimate for the solution to the forward problem is derived.

The rest of the paper is organized as follows. In Sect. 2 we formulate the forward and inverse problems, which are studied, and give some definitions. In Sect. 3 the subordination principle is presented and the injectivity of the subordination integral transform is established. Based on these results, in Sect. 4 the uniqueness for the coefficient inverse problem is proved. In Sect. 5 the eigenfunction expansion of the solution to the forward problem is derived in terms of multinomial Mittag-Leffler functions and a regularity estimate is established by applying the subordination relation. The last Sect. 6 contains concluding remarks.

2 Problem formulation and definitions

Assume $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, is a bounded domain with sufficiently smooth boundary $\partial\Omega$. In this work the following equation is considered

$$D_t^{\alpha} u(x,t) + \sum_{j=1}^{m} q_j D_t^{\alpha_j} u(x,t) = \Delta u(x,t) + p(x)u(x,t), \quad x \in \Omega,\ t > 0, \tag{2.1}$$

where Δ is the Laplace operator acting with respect to the spatial variables and D_t^{α}, $D_t^{\alpha_j}$ are fractional time-derivatives in the Caputo sense.

Recall that the fractional Caputo derivative D_t^{α} of order $\alpha > 0$ is defined as follows [2, 24]

$$D_t^{\alpha} f(t) = \frac{d^n}{dt^n} \int_0^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} \left(f(\tau) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{\tau^k}{k!} \right) d\tau,$$

where n is a positive integer, such that $n - 1 < \alpha < n$.

We suppose that the parameters in equation (2.1) obey the following restrictions

$$\begin{aligned} &\alpha \in (1,2), \quad \alpha > \alpha_1 > \cdots > \alpha_m > 0, \quad \alpha - \alpha_m \le 1, \\ &q_j > 0, \quad j = 1, \cdots, m. \end{aligned} \tag{2.2}$$

Consider the initial-boundary-value problem (IBVP) for equation (2.1) with the following initial and boundary conditions

$$\begin{aligned} &u(x,0) = a(x), \ \lim_{t \to 0^+} \frac{\partial}{\partial t} u(x,t) = 0, \quad x \in \Omega, \\ &u(x,t) = 0, \ \ x \in \partial\Omega,\ t > 0. \end{aligned} \tag{2.3}$$

The IBVP (2.1)–(2.3) (in a more general inhomogeneous form and without the assumption $\alpha - \alpha_m \le 1$) is studied recently in [34]. Among other results, it is established in [34] that under the conditions $p \in C(\overline{\Omega})$, $p \le 0$ in $\overline{\Omega}$, and $a \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists a unique solution to IBVP (2.1)–(2.3), satisfying

$$u \in C([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega)), \tag{2.4}$$

where $H_0^1(\Omega)$ and $H^2(\Omega)$ are standard notations for Sobolev spaces (see e.g. [1]). This is an extension of a result in the seminal work [30], concerning the single-term fractional diffusion-wave equation (a particular case of problem (2.1)–(2.3), where $m = 0$).

The function $p(x)$ describes a spatial diffusivity coefficient and, from a physical point of view, it is important how to choose $p(x)$, so that the solution behaves appropriately. A standard inverse problem, referred to as coefficient inverse problem, is to recover the coefficient $p(x)$ from a combination of initial data and over-specified boundary data.

The main purpose of this paper is to prove the uniqueness for the coefficient inverse problem of determining $p(x)$, $x \in \Omega$, by interior data $u|_{\omega\times(0,T)}$, where $T > 0$ is given and $\omega \subset \Omega$ is a suitable subdomain. Such a result for the particular single-term case of IBVP (2.1)–(2.3) with $m = 0$ is deduced in [25], see also [22], by applying the subordination principle with respect to a second order equation and the analogous uniqueness result for this equation. Inspired by the idea of this work, our proof of the uniqueness for the coefficient inverse problem for equation (2.1) is based on the result in [25] and the injectivity property of the subordination relation between the solutions of the multi-term equation and the single-term one.

Let $L^2(\Omega)$ be a standard L^2-space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Assume $p \in W^{1,\infty}(\Omega)$ and $p \le 0$ in $\overline{\Omega}$. Let us define an operator A_p in the space $L^2(\Omega)$ as follows

$$(A_p y)(x) = \Delta y(x) + p(x)y(x), \quad x \in \Omega; \quad D(-A_p) = H^2(\Omega) \cup H_0^1(\Omega), \tag{2.5}$$

where $H^2(\Omega)$, $H_0^1(\Omega)$ denote Sobolev spaces.

Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the set of all eigenvalues of the operator $-A_p$. It is known that the eigenvalues are positive and with finite multiplicity, $0 < \lambda_1 \le \lambda_2 \le \cdots$, $\lambda_k \to \infty$ as $k \to \infty$. Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be the corresponding eigenfunctions:

$$A_p\varphi_k = -\lambda_k\varphi_k, \quad \varphi_k \in H^2(\Omega) \cup H_0^1(\Omega).$$

The eigenfunctions form an orthonormal basis of $L^2(\Omega)$.

The operator $(-A_p)^\gamma$, $\gamma > 0$, is defined as follows

$$D\left((-A_p)^\gamma\right) = \left\{ f \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\gamma} |(f,\varphi_k)|^2 < \infty \right\},$$

$$(-A_p)^\gamma f = \sum_{k=1}^{\infty} \lambda_k^{\gamma}(f,\varphi_k)\varphi_k$$

and $D\left((-A_p)^\gamma\right)$ is a Hilbert space with the norm

$$\|f\|_{D((-A_p)^\gamma)} = \|(-A_p)^\gamma f\| = \left(\sum_{k=1}^{\infty} \lambda_k^{2\gamma} |(f,\varphi_k)|^2 \right)^{1/2}.$$

3 Subordination principle

Along with IBVP (2.1)–(2.3) let us consider the particular case of the single-term equation, subject to the same initial and boundary conditions

$$
\begin{aligned}
&D_t^\alpha v(x,t) = \Delta v(x,t) + p(x)v(x,t), \quad x \in \Omega,\ t > 0,\\
&v(x,0) = a(x), \ \lim_{t\to 0^+} \frac{\partial}{\partial t} v(x,t) = 0, \quad x \in \Omega, \\
&v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0.
\end{aligned} \tag{3.1}
$$

We use the subordination principle, discussed in [6] and [8], which relates the solution of problem (2.1)–(2.3) for the multi-term equation to that of problem (3.1). For the formulation of the subordination results we use the terminology of abstract evolution equations (for basic notions see e.g. [6]).

Rewriting IBVP (2.1)–(2.3) in abstract form, we are concerned with the following Cauchy problem for the multi-term fractional evolution equation

$$
D_t^\alpha u(t) + \sum_{j=1}^{m} q_j D_t^{\alpha_j} u(t) = A_p u(t), \quad t > 0; \quad u(0) = a,\ u'(0) = 0, \tag{3.2}
$$

where $a \in L^2(\Omega)$ and the parameters α, α_j, q_j satisfy restrictions (2.2).

Correspondingly, problem (3.1) in abstract form reads

$$
D_t^\alpha v(t) = A_p v(t), \quad t > 0; \quad v(0) = a,\ v'(0) = 0, \tag{3.3}
$$

where $a \in L^2(\Omega)$ and $1 < \alpha < 2$.

The operator A_p, defined in (2.5), is a closed linear densely defined operator, that generates a bounded cosine family $S_2(t)$, (see e.g. [1], Section 7.2.), defined by the following eigenfunction decomposition

$$
S_2(t)a = \sum_{k=1}^{\infty} (a, \varphi_k) \cos\left(\sqrt{\lambda_k} t\right) \varphi_k(x).
$$

Recall that the function $w(x,t) = S_2(t)a$ is a solution of the second-order equation $w_{tt} = \Delta w + p(x)w$ with initial and boundary conditions (2.3).

Moreover, it is known that the operator A_p generates a bounded solution operator $S_\alpha(t)$ to Cauchy problem (3.3), defined by the eigenfunction expansion

$$
S_\alpha(t)a = \sum_{k=1}^{\infty} (a, \varphi_k) E_\alpha(-\lambda_k t^\alpha) \varphi_k(x),
$$

where $E_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$ denotes the Mittag-Leffler function.

The unique solution of IBVP (3.1) is then given by $v(x,t) = S_\alpha(t)a$.

For establishing a subordination relation it is convenient to rewrite problem (3.2) as an abstract Volterra integral equation and use the general subordination theorems in [7]. Applying Laplace transform

$$\mathcal{L}\{f(t)\}(s) = \widehat{f}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$$

we obtain from (3.2) by the use of the fundamental formula for Caputo derivatives

$$\mathcal{L}\{D_t^\beta f\}(s) = s^\beta \widehat{f}(s) - \sum_{k=0}^{n-1} f^{(k)}(0) s^{\beta-1-k}, \quad n-1 < \beta \le n, \ n \in \mathbb{N}, \tag{3.4}$$

the abstract Volterra integral equation

$$u(t) = a + \int_0^t k(t-\tau) A_p u(\tau)\, d\tau, \tag{3.5}$$

where the scalar kernel $k(t)$ is defined by its Laplace transform

$$\widehat{k}(s) = 1/g(s) \tag{3.6}$$

with $g(s)$ being the following characteristic function

$$g(s) = s^\alpha + \sum_{j=1}^m q_j s^{\alpha_j}, \quad s > 0. \tag{3.7}$$

3.1 Properties of the characteristic function $g(s)$

To prove subordination principle we need some facts from the theory of Bernstein functions. For a concise overview of the special classes of functions related to Bernstein functions we refer to [33], a short list of basic properties can be found e.g. in [6]. Denote by $\mathcal{CBF}$ the set of complete Bernstein functions. A function $\varphi : (0,\infty) \to \mathbb{R}$ is said to be a complete Bernstein functions if and only if $\varphi(s)/s$ can be written as a restriction of the Laplace transform of a completely monotone function to the real positive semi-axis. Recall that $f : (0,\infty) \to \mathbb{R}$ is completely monotone if it is of class C^∞ and

$$(-1)^n f^{(n)} \ge 0, \quad n = 0, 1, 2, \ldots \tag{3.8}$$

A non-negative function $\varphi \in C^\infty(0,\infty)$ is said to be a Bernstein function if its first derivative φ' is completely monotone. Any complete Bernstein function is a Bernstein function.

Define the following set of functions

$$\mathcal{CBF}^\delta := \left\{\varphi^\delta : \varphi \in \mathcal{CBF}\right\}, \quad \delta > 0. \tag{3.9}$$

The inclusion holds true [7]

$$\mathcal{CBF}^{\delta_0} \subseteq \mathcal{CBF}^{\delta}, \quad 0 < \delta_0 \leq \delta. \tag{3.10}$$

A property of the function $g(s)$, which plays a basic role in our considerations, is established next.

Proposition 1 *Under the assumptions* (2.2) *the characteristic function* $g(s)$*, defined in* (3.7)*, satisfies the property*

$$g(s) \in \mathcal{CBF}^{\alpha}. \tag{3.11}$$

Proof This property was established earlier in [8], see Proposition 4.1. Here we give a different proof, based on the decomposition

$$g(s) = s^{\beta}\left(s^{\alpha-\beta} + \sum_{j=1}^{m} q_j s^{\alpha_j-\beta}\right),$$

where $\beta = \min\{\alpha_m, 1\}$. Let us note first that $s^{\delta} \in \mathcal{CBF}^{\delta}$ for any $\delta > 0$, since $s \in \mathcal{CBF}$. Moreover, taking into account that $0 \leq \alpha_j - \beta \leq \alpha - \beta \leq 1$, Proposition 2 in [7] implies

$$s^{\alpha-\beta} + \sum_{j=1}^{m} q_j s^{\alpha_j-\beta} \in \mathcal{CBF}^{\alpha-\beta}.$$

Therefore, $g(s)$ is a product of two functions from the sets $\mathcal{CBF}^{\beta}$ and $\mathcal{CBF}^{\alpha-\beta}$, respectively. Then the product property $\mathcal{CBF}^{\delta_1}.\mathcal{CBF}^{\delta_2} \subseteq \mathcal{CBF}^{\delta_1+\delta_2}$ (for the proof see [7], Proposition 1) implies $g(s) \in \mathcal{CBF}^{\alpha}$. □

Remark 1 The assumption $\alpha - \alpha_m \leq 1$ is necessary for the property (3.11), as well as for the weaker property $g(s) \in \mathcal{CBF}^{2}$, as the following counterexample shows.

Consider the simple characteristic function $g_\alpha(s) = s^\alpha + 1$, where $1 < \alpha < 2$. First, we show that $g_\alpha(s) \notin \mathcal{CBF}^{\alpha}$ by proving that the function $h(s) = (g_\alpha(s))^{1/\alpha}$ is not concave for $s > 0$. Indeed, since $\alpha > 1$, the second derivative of $h(s)$ satisfies for $s > 0$

$$h''(s) = (\alpha - 1)s^{\alpha-2}\left(s^\alpha + 1\right)^{1/\alpha-2} > 0.$$

Therefore, concavity is not satisfied, which means that $h(s)$ is not a Bernstein function. In a similar way we show that $g_\alpha(s) \notin \mathcal{CBF}^{2}$, taking $h(s) = (g_\alpha(s))^{1/2}$. In this case the second derivative is given by the expression

$$h''(s) = \frac{\alpha s^{\alpha-2}\left(2(\alpha-1) - (2-\alpha)s^{\alpha}\right)}{4\left(s^\alpha + 1\right)^{3/2}}, \quad s > 0,$$

which implies

$$h''(s) > 0 \text{ for } s \in (0, M_\alpha), \quad M_\alpha = \left(\frac{2(\alpha-1)}{2-\alpha}\right)^{1/\alpha} > 0.$$

Therefore, $h(s)$ is not concave on the whole half-line $(0, \infty)$ and, thus, it can not be a Bernstein function.

3.2 Subordination relation and its injectivity

Applying the general subordination theorems in [7] (Theorems 4 and 5), we deduce the following subordination result:

Theorem 1 *Under the assumptions* (2.2) *the IBVP* (2.1)–(2.3) *admits a bounded analytic solution operator* $S(t)$*, which is related to* $S_\alpha(t)$ *via the subordination identity*

$$S(t) = \int_0^\infty \Phi(t,\tau) S_\alpha(\tau)\, d\tau, \quad t > 0. \tag{3.12}$$

The kernel $\Phi(t,\tau)$ *is defined via the Laplace transform pair*

$$\int_0^\infty e^{-st} \Phi(t,\tau)\, dt = \frac{g(s)^{1/\alpha}}{s} \exp\left(-\tau g(s)^{1/\alpha}\right), \quad \tau > 0, \tag{3.13}$$

where $g(s)$ *is given in* (3.7)*. Moreover,*

$$\Phi(t,\tau) \geq 0, \quad \int_0^\infty \Phi(t,\tau)\, d\tau = 1, \qquad t, \tau > 0. \tag{3.14}$$

Theorem 1 implies that the unique solution of IBVP (2.1)–(2.3) admits the representation

$$u(x,t) = S(t)a = \int_0^\infty \Phi(t,\tau) S_\alpha(\tau) a\, d\tau = \int_0^\infty \Phi(t,\tau) v(x,\tau)\, d\tau, \tag{3.15}$$

where $v(x,t)$ is the solution to IBVP (3.1). Moreover, analyticity of the solution operator $S(t)$ implies that $u(x,t)$ is a real analytic function in the variable t, $t > 0$.

The subordination relation (3.15) between the solution of IBVP (2.1)–(2.3) for the multi-term equation and its simpler single-term version IBVP (3.1) allows some properties of IBVP (2.1)–(2.3) to be derived from the analogous properties of the simpler problem (3.1).

The subordination identity (3.12) defines an integral transform with kernel $\Phi(t,\tau)$. Next we establish the uniqueness property of this integral transform.

Theorem 2 *Let $f(t)$, $t \geq 0$, be a function with values in $L^2(\Omega)$, such that $\|f\| \leq M$, $t \geq 0$. Then the integral transform*

$$\widetilde{f}(t) = \int_0^\infty \Phi(t,\tau) f(\tau)\, d\tau, \quad t > 0, \tag{3.16}$$

with kernel Φ satisfying (3.13) is well defined. Moreover, if $\widetilde{f}(t) = 0$ for a.e. $t > 0$ then $f(t) = 0$ for a.e. $t > 0$.

Proof The properties (3.14) of the kernel $\Phi(t,\tau)$ and the boundedness of $f(t)$ imply

$$\int_0^\infty \|\Phi(t,\tau) f(\tau)\|\, d\tau \leq M \int_0^\infty \Phi(t,\tau)\, d\tau = M, \quad t > 0.$$

Therefore, the integral in (3.16), defining $\widetilde{f}$, converges.

For $s > 0$ let us apply the Laplace transform to both sides of identity (3.16). We can interchange the order of integration and obtain by applying (3.13) the following representation of the Laplace transform of the function $\widetilde{f}$

$$\int_0^\infty e^{-st} \widetilde{f}(t)\, dt = \frac{g(s)^{1/\alpha}}{s} \int_0^\infty \exp\left(-\tau g(s)^{1/\alpha}\right) f(\tau)\, d\tau.$$

Since $g(s)^{1/\alpha}$ is a complete Bernstein function, it is in particular continuous and monotonically non-decreasing for $s > 0$. Moreover, $g(0) = 0$ and $g(+\infty) = +\infty$. Therefore, the assumption $\widetilde{f}(t) = 0$ for a.e. $t > 0$, implies that the Laplace transform of f vanishes on the right half of the real line. Then the uniqueness property of Laplace transform (see e.g. [1], Theorem 1.7.3) implies $f = 0$ for a.e. $t > 0$. □

4 Uniqueness for the coefficient inverse problem

In [25] the uniqueness for a coefficient inverse problem of determining the function $p(x)$ in (3.1) is established under appropriate additional data. In this section we use the uniqueness result of [25] and the properties of the subordination relation from the previous section to extend this result to the multi-term case. In this work we make the same assumptions as in [25] in order to guarantee the uniqueness result for the coefficient inverse problem for the single-term equation (3.1).

Let ω be a sub-domain of Ω, such that $\partial\omega \supset \partial\Omega$, and set

$$\mathcal{U}_M = \left\{ p \in W^{1,\infty}(\Omega) : \ p \leq 0 \text{ in } \Omega, \ \|p\|_{W^{1,\infty}(\Omega)} \leq M, \ p|_\omega = \eta \right\}, \tag{4.1}$$

where $M > 0$ is a constant and η is a smooth function, which are arbitrarily chosen.

Let us denote by $u_p(x,t)$ the solution to problem (2.1)–(2.3) and by $v_p(x,t)$ the solution to problem (3.1).

Theorem 3 *Assume conditions* (2.2) *are satisfied,*

$$a \in H^3(\Omega) \cup H_0^2(\Omega), \quad \Delta a \in H_0^1(\Omega), \quad a(x) > 0 \, for \; x \in \overline{\Omega \backslash \omega},$$

and let $p, q \in \mathcal{U}_M$, *where the set of functions* $\mathcal{U}_M$ *is defined in* (4.1). *If*

$$u_p(x,t) = u_q(x,t), \quad x \in \omega, \; t \in (0,T), \tag{4.2}$$

then $p(x) = q(x)$ *for* $x \in \Omega$.

Proof It is proven in [25] that, under the assumptions of the theorem, $v_p = v_q$ in $\omega \times (0,T)$ implies $p = q$ in Ω. For the proof of the theorem we use this fact and the results from the previous section. Theorem 1 yields the following subordination relations for $x \in \Omega$ and $t > 0$

$$u_p(x,t) = \int_0^\infty \Phi(t,\tau) v_p(x,\tau)\, d\tau,$$
$$u_q(x,t) = \int_0^\infty \Phi(t,\tau) v_q(x,\tau)\, d\tau.$$

The analyticity of the solution operator $S(t)$ in Theorem 1 implies analyticity in t of the solutions $u_p(x,t)$ and $u_q(x,t)$. Therefore, assumption (4.2) implies $u_p(x,t) = u_q(x,t)$ for $(x,t) \in \omega \times (0,\infty)$, i.e.

$$\int_0^\infty \Phi(t,\tau) \left(v_p(x,\tau) - v_q(x,\tau) \right) d\tau = 0, \quad x \in \omega, \quad t > 0.$$

For arbitrary $x \in \omega$, Theorem 2 yields $v_p(x,t) = v_q(x,t)$ for $t > 0$. Applying the theorem in [25], concerning the single-term version of the present theorem, we deduce $p(x) = q(x)$, $x \in \Omega$. $\square$

5 A regularity estimate

In this section we apply the subordination principle in Theorem 1 to obtain some regularity estimates for the solution of the forward problem (2.1)–(2.3). Regularity estimates for the multi-term equation on bounded domain have been derived so far based on the eigenfunction expansion of the solution, see [34], which relies on estimates for the multinomial Mittag-Leffler function.

5.1 Eigenfunction expansion

First, we present briefly the derivation of the unique solution to IBVP (2.1)–(2.3) by the Fourier method. The corresponding equations in the eigenspaces

$$D_t^\alpha y_k(t) + \sum_{j=1}^{m} q_j D_t^{\alpha_j} y_k(t) + \lambda_k y_k(t) = 0, \quad y_k(0) = 1, y_k'(0) = 0, \tag{5.1}$$

can be solved by applying Laplace transform. Using the property (3.4) we obtain

$$\widehat{y_k}(s) = \frac{g(s)}{s(g(s)+\lambda_k)}, \tag{5.2}$$

where the function $g(s)$ is defined in (3.7).

To obtain from (5.2) an explicit representation for the function $y(t)$, let us consider the multinomial Mittag-Leffler function

$$E_{(\mu_1,\ldots,\mu_n),\beta}(z_1,\ldots,z_n) = \sum_{k=0}^{\infty} \sum_{\substack{k_1+\ldots+k_n=k \\ k_1\geq 0,\ldots,k_n\geq 0}} \frac{k!}{k_1!\ldots k_n!} \frac{\prod_{j=1}^n z_j^{k_j}}{\Gamma\left(\beta + \sum_{j=1}^n \mu_j k_j\right)},$$

which is introduced in [13] as a generalization of the classical Mittag-Leffler function and used for solving multi-term fractional differential equations with constant coefficients, see e.g. [5, 23, 34]. For more details on Mittag-Leffler functions and their generalizations we refer to [10], see also the very recent papers [18, 19, 26, 27].

Taking into account the Laplace transform pair (see e.g. [5])

$$\mathcal{L}\left\{t^{\beta-1} E_{(\mu_1,\ldots,\mu_n),\beta}(-a_1 t^{\mu_1},\ldots,-a_n t^{\mu_n})\right\}(s) = \frac{s^{-\beta}}{1+\sum_{j=1}^m a_j s^{-\mu_j}} \tag{5.3}$$

we obtain from (5.2) that the solution $y_k(t)$ of (5.1) can be expressed in terms of the multinomial Mittag-Leffler function as follows

$$y_k(t) = 1 - \lambda_k t^\alpha E_{(\alpha,\alpha-\alpha_1,\ldots,\alpha-\alpha_m),\alpha+1}\left(-\lambda_k t^\alpha, -q_1 t^{\alpha-\alpha_1},\ldots,-q_m t^{\alpha-\alpha_m}\right). \tag{5.4}$$

Therefore, the unique solution of IBVP (2.1)–(2.3) admits the eigenfunction expansion

$$u(x,t) = \sum_{k=1}^{\infty} (a,\varphi_k) y_k(t) \varphi_k(x), \tag{5.5}$$

where $y_k(t)$ is given in (5.4). The eigenfunction expansion (5.5) can serve as a starting point for obtaining regularity estimates for the solution of IBVP (2.1)–(2.3) by using some properties of the multinomial Mittag-Leffler function, see [34]. Next, we give an example of deriving regularity estimates by the use of the subordination principle and known estimates for the single-term equation.

5.2 Application of subordination principle

First, let us mention the simple estimate for the solution of the single-term diffusion-wave equation (3.1): $\|v(\cdot,t)\| \le C\|a\|$. Applying the subordination relation (3.15) and, taking into account the properties (3.14) of the subordination function $\Phi(t,\tau)$, we derive the same estimate for the solution $u(x,t)$ of the IBVP (2.1)–(2.3) for the multi-term equation:

$$\|u(\cdot,t)\| \le \int_0^\infty \Phi(t,\tau)\|v(\cdot,\tau)\|\,d\tau \le C\int_0^\infty \Phi(t,\tau)\,d\tau\|a\| = C\|a\|.$$

By applying a similar idea, next we generalize the following estimate for the single-term diffusion-wave equation (3.1) when $a \in D\left((-A_p)^\gamma\right)$, which is obtained in [15] (in different notations and for $p \equiv 0$, but the proof can be easily extended to our case):

$$\|v(\cdot,t)\|_{D((-A_p)^\beta)} \le Ct^{-\alpha(\beta-\gamma)}\|a\|_{D((-A_p)^\gamma)}, \quad t > 0, \tag{5.6}$$

where $0 \le \gamma \le \beta \le 1$ and $\alpha \in (1,2)$.

Theorem 4 *Let assumptions* (2.2) *be satisfied and* $a \in D\left((-A_p)^\gamma\right)$ *for some* $\gamma \in (0,1)$. *Then for any* $\beta \in (0,1)$*, such that* $0 \le \beta-\gamma \le 1/2$*, the solution to IBVP* (2.1)–(2.3) *satisfies the estimate*

$$\|u(\cdot,t)\|_{D((-A_p)^\beta)} \le C_T t^{-\alpha(\beta-\gamma)}\|a\|_{D((-A_p)^\gamma)}, \quad t \in (0,T). \tag{5.7}$$

Proof Applying the subordination relation (3.15), properties (3.14), and estimate (5.6), we derive

$$\begin{aligned}\|u(\cdot,t)\|_{D((-A_p)^\beta)} &\le \int_0^\infty \Phi(t,\tau)\|v(\cdot,\tau)\|_{D((-A_p)^\beta)}\,d\tau \\ &\le C\int_0^\infty \Phi(t,\tau)\tau^{-\alpha(\beta-\gamma)}\,d\tau\|a\|_{D((-A_p)^\gamma)}.\end{aligned} \tag{5.8}$$

Denote $I(t) := \int_0^\infty \Phi(t,\tau)\tau^{-\alpha(\beta-\gamma)}\,d\tau$. Since $0 \le \alpha(\beta-\gamma) \le 1$ the integral is convergent and the function $I(t)$ is well defined. Applying the Laplace transform pair (3.13) and the following identity for the Gamma function

$$\int_0^\infty x^{p-1}\exp(-qx)\,dx = \Gamma(p)q^{-p}, \quad p>0,\ q>0,$$

we obtain for the Laplace transform $\widehat{I}(s)$ of the function $I(t)$

$$\begin{aligned}\widehat{I}(s) &= \frac{g(s)^{1/\alpha}}{s}\int_0^\infty \tau^{-\alpha(\beta-\gamma)}\exp(-\tau g(s)^{1/\alpha})\,d\tau \\ &= \Gamma\left(1-\alpha(\beta-\gamma)\right)\frac{g(s)^{\beta-\gamma}}{s},\end{aligned} \tag{5.9}$$

where the function $g(s)$ is defined in (3.7). Therefore

$$\widehat{I}(s) = \Gamma\left(1-\alpha(\beta-\gamma)\right) s^{\alpha(\beta-\gamma)-1}\left(1+\sum_{j=1}^{m} q_j s^{\alpha_j-\alpha}\right)^{\beta-\gamma}. \tag{5.10}$$

From (5.10) we find explicit expression for the function $I(t)$ in terms of multinomial Prabhakar functions [5]

$$E^{\delta}_{(\mu_1,\ldots,\mu_n),\beta}(z_1,\ldots,z_n) = \sum_{k=0}^{\infty} \sum_{\substack{k_1+\ldots+k_n=k \\ k_1\geq 0,\ldots,k_n\geq 0}} \frac{(\delta)_k}{k_1!\ldots k_n!} \frac{\prod_{j=1}^{n} z_j^{k_j}}{\Gamma\left(\beta+\sum_{j=1}^{n}\mu_j k_j\right)},$$

where $(\delta)_k$ denotes the Pochhammer symbol

$$(\delta)_k = \delta(\delta+1)\ldots(\delta+k-1), \quad k\in\mathbb{N},\ \delta\in\mathbb{R}; \quad (\delta)_0 = 1,\ \delta\in\mathbb{R}\backslash\{0\}.$$

The Laplace transform pair [5]

$$\mathcal{L}\left\{t^{\beta-1}E^{\delta}_{(\mu_1,\ldots,\mu_n),\beta}(-a_1 t^{\mu_1},\ldots,-a_n t^{\mu_n})\right\}(s) = s^{-\beta}\left(1+\sum_{j=1}^{m} a_j s^{-\mu_j}\right)^{-\delta}$$

and (5.10) imply

$$I(t) = c_{\alpha,\beta,\gamma}\, t^{-\alpha(\beta-\gamma)} E^{\gamma-\beta}_{(\alpha-\alpha_1,\ldots,\alpha-\alpha_m),1-\alpha(\beta-\gamma)}\left(-q_1 t^{\alpha-\alpha_1},\ldots,-q_m t^{\alpha-\alpha_m}\right),$$

where $c_{\alpha,\beta,\gamma} = \Gamma\left(1-\alpha(\beta-\gamma)\right)$. Plugging this result in (5.8) and taking into account the asymptotic expansion of the multinomial Prabhakar function for small t, given by the first term in the series expansion, we derive (5.7). □

6 Concluding remarks

In this work we established a uniqueness result for a coefficient inverse problem for the multi-term time-fractional wave equation on a bounded domain. The proof is based on the subordination principle and the uniqueness result for the particular case of the single-term equation.

Applying the subordination principle, we can easily obtain some regularity estimates for the multi-term wave equation, as the presented example demonstrates. However, the applicability of this method is restricted. For instance, in the considered in Theorem 4 example, the power of time variable in the original estimate (5.6) should be such that, after plugging in the subordination identity, the convergence of the obtained integral (5.8) is guaranteed.

Other types of inverse problems, e.g. the unique determination of the orders of fractional derivatives under some additional data, can be studied by the use of the eigenfunction expansion (5.5) and relevant properties of the multinomial Mittag-Leffler function. In this way the results in [21, 25] concerning the unique determination of the fractional orders in the single-term case and the multi-term diffusion case, respectively, can be extended to the considered in the present work multi-term wave equation.

Declarations

Conflict of interest The author declares no conflict of interest.

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